

# On McKay's propagation theorem for the Foulkes conjecture

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September 17, 2015

## Abstract

We translate the main theorem in Tom McKay's paper "On plethysm conjectures of Stanley and Foulkes" (J. Alg. 319, 2008, pp. 2050–2071) to the language of weight spaces and projections onto invariant spaces of tensors, which makes its proof short and elegant.

Keywords: Representation theory of the symmetric group; Plethysm; Foulkes Conjecture; Foulkes-Howe Conjecture

MSC2010: 20C30

## 1 Introduction

The Foulkes conjecture [Fou50] states an inequality of certain representation theoretic multiplicities. It comes in several different equivalent formulations, the most straightforward one being the following.

**1.1 Conjecture** (Foulkes conjecture). *Let  $a, b \in \mathbb{N}_{>0}$  with  $a \leq b$ . Let  $U$  be a finite dimensional vector space of dimension at least  $b$ . Then for every partition  $\lambda$  the multiplicity of the irreducible  $\mathrm{GL}(U)$  representation  $\{\lambda\}$  in the plethysm  $\mathrm{Sym}^a(\mathrm{Sym}^b U)$  is at most as large as the multiplicity of  $\{\lambda\}$  in  $\mathrm{Sym}^b(\mathrm{Sym}^a U)$ .*

The inequality  $a \leq b$  is important: We know that  $\mathrm{Sym}^a(\mathrm{Sym}^b U)$  contains irreducible  $\mathrm{GL}(U)$  representations with up to  $a$  parts, but  $\mathrm{Sym}^b(\mathrm{Sym}^a U)$  contains irreducible  $\mathrm{GL}(U)$  representations with up to  $b$  parts.

Using Schur-Weyl duality [Gay76] (see also [Ike12]) one can interpret Conjecture 1.1 in terms of representations of the symmetric group and we will use that interpretation later in this paper.

Conjecture 1.1 is true for  $a \leq 5$ : for  $a \leq 2$  see the explicit formulas in [Thr42], for the case  $a \leq 3$  see [DS00], and see Corollary 1.4 for  $a \leq 4$  and Corollary 1.5 for  $a \leq 5$ . Brion [Bri93] showed that Conjecture 1.1 is true in the cases where  $b$  is large enough with respect to  $a$ . Conjecture 1.1 is true if we only

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consider partitions  $\lambda$  with at most 2 rows [Her54], a phenomenon called Hermite reciprocity. Manivel showed that Conjecture 1.1 is also true for all partitions with very long first rows [Man98, Thm. 4.3.1]. In this case we do not only have an inequality of the multiplicities, but equality.

Conjecture 1.1 is equivalent to saying that there exists a  $\mathrm{GL}(U)$  equivariant inclusion map

$$\mathrm{Sym}^a(\mathrm{Sym}^b U) \hookrightarrow \mathrm{Sym}^b(\mathrm{Sym}^a U). \quad (1.2)$$

A natural candidate for the map in (1.2) was the following map  $\Psi_{a \times b}$ :

$$\begin{array}{ccc} \mathrm{Sym}^a(\mathrm{Sym}^b U) & \xrightarrow{\Psi_{a \times b}} & \mathrm{Sym}^b(\mathrm{Sym}^a U) \\ \downarrow \iota & & \uparrow \varrho \\ \otimes^a(\otimes^b U) & \xrightarrow{r} & \otimes^b(\otimes^a U) \end{array}$$

where  $\iota$  denotes the canonical embedding of symmetric tensors in the space of all tensors,  $\varrho$  is the canonical projection, and  $r$  is the canonical isomorphism given by reordering tensor factors. This map has a natural analog  $\psi_{a \times b}$  in the symmetric group interpretation that we will discuss in Section 2. It was combinatorially defined in [BL89].

Hadamard conjectured [Had97] that  $\Psi_{a \times b}$  is injective for all  $a \leq b$ . Howe [How87, p. 93] wrote that this “is reasonable to expect”. However, Müller and Neunhöffer [MN05] showed that  $\Psi_{5 \times 5}$  has a nontrivial kernel. In [CIM15] the kernel of  $\Psi_{5 \times 5}$  is determined as a  $\mathrm{GL}(U)$  representation: It is multiplicity free and consists of the following types of irreducible representations:  $(14, 7, 2, 2)$ ,  $(13, 7, 2, 2, 1)$ ,  $(12, 7, 3, 2, 1)$ ,  $(12, 6, 3, 2, 2)$ ,  $(12, 5, 4, 3, 1)$ ,  $(11, 5, 4, 4, 1)$ ,  $(10, 8, 4, 2, 1)$ , and  $(9, 7, 6, 3)$ . Also it is shown in [CIM15] that  $\Psi_{6 \times 6}$  is not injective.

McKay [McK08] contributed the following main theorem.

**1.3 Theorem** (McKay’s main theorem). *If  $\Psi_{a \times (b-1)}$  is injective, then  $\Psi_{a \times c}$  is injective for all  $c \geq b$ .*

The proof uses the analog map  $\psi_{a \times b}$ , defined in Section 2, and decomposes it into a composition of two maps (Section 3) whose injectivity is proved independently (Sections 4 and Sections 5).

Theorem 1.3 allows us to verify Conjecture 1.1 in infinitely many cases while only doing a finite calculation: [MN05] calculate that  $\Psi_{4 \times 4}$  is injective, so Theorem 1.3 implies the following corollary.

**1.4 Corollary.** *Conjecture 1.1 is true in all cases where  $a = 4$ .*

Although  $\Psi_{5 \times 5}$  is not injective, if  $a = b$  then  $\mathrm{Sym}^a(\mathrm{Sym}^b U) = \mathrm{Sym}^b(\mathrm{Sym}^a U)$ . The recent calculation [CIM15] reveals that  $\Psi_{5 \times 6}$  is injective, therefore Theorem 1.3 implies the following corollary.

**1.5 Corollary.** *Conjecture 1.1 is true in all cases where  $a = 5$ .*

**1.6 Remark.** Let  $\text{Ch}_b := \overline{\text{GL}_b(X_1 X_2 \cdots X_b)} \subseteq \text{Sym}^b \mathbb{C}^b$  denote the  $\text{GL}_b$  orbit closure of the monomial  $X_1 X_2 \cdots X_b$ . We call this orbit closure the  $b$ th Chow variety. The kernel of  $\Psi_{a \times b}$  is known to be the homogeneous degree  $a$  part of the vanishing ideal of  $\text{Ch}_b$ , as was shown by Hadamard, see e.g. [Lan11, Section 8.6].

For more information on the history of the Foulkes conjecture and the kernel of  $\Psi_{a \times b}$  we refer the interested reader to [Lan15, Section 7.1].

## Acknowledgments

I thank JM Landsberg for bringing McKay's paper to my attention, for valuable discussions, and for providing help with the historical background.

## 2 Preliminaries

Fix natural numbers  $a, b \in \mathbb{N}_{>0}$  and let  $V := \mathbb{C}^a \oplus \mathbb{C}^b$ . The group  $\text{GL}_a \times \text{GL}_b$  acts canonically on  $\mathbb{C}^a \oplus \mathbb{C}^b$  and hence on the  $ab$ th tensor power  $\bigotimes^{ab} V$ . Let  $\mathfrak{S}_a$  be the symmetric group on  $a$  letters and embed  $\mathfrak{S}_a \subseteq \text{GL}_a$  via permutation matrices. Let  $a \times b := (b, b, \dots, b)$  denote the partition of  $ab$  whose Young diagram is a rectangle with  $a$  rows and  $b$  columns (in anglophone notation). Let  $\emptyset$  denote the empty partition, i.e.,  $\emptyset := 0 \times 0$ . For complex numbers  $s_1, \dots, s_a$  let  $\text{diag}(s_1, \dots, s_a)$  denote the diagonal matrix with  $s_1, \dots, s_a$  on the main diagonal. Analogously for  $\text{diag}(t_1, \dots, t_b)$ . For  $\alpha \in \mathbb{N}^a$  and  $\beta \in \mathbb{N}^b$  the set of tensors  $(\bigotimes^{ab} V)_{\alpha, \beta} :=$

$$\{w \in \bigotimes^{ab} V \mid (\text{diag}(s_1, \dots, s_a), \text{diag}(t_1, \dots, t_b))w = s_1^{\alpha_1} \cdots s_a^{\alpha_a} t_1^{\beta_1} \cdots t_b^{\beta_b} w\}$$

is called the  $(\alpha, \beta)$  weight space of  $\bigotimes^{ab} V$ . Here  $\alpha$  and  $\beta$  might be partitions, but could also be *weak compositions*, i.e., we do not require the entries of  $\alpha \in \mathbb{N}_{\geq 0}^a$  and  $\beta \in \mathbb{N}_{\geq 0}^b$  to be ordered. The weight space  $(\bigotimes^{ab} V)_{a \times b, \emptyset}$  is closed under the action of  $\mathfrak{S}_a$ . Let  $(\bigotimes^{ab} V)_{a \times b, \emptyset}^{\mathfrak{S}_a}$  denote the  $\mathfrak{S}_a$  invariant space. Analogously define  $(\bigotimes^{ab} V)_{\emptyset, b \times a}^{\mathfrak{S}_b}$ . On the space of tensors  $\bigotimes^{ab} V$  we have the canonical action of  $\mathfrak{S}_{ab}$  via permutation of the tensor factors. To avoid confusion with  $\mathfrak{S}_a$  or  $\mathfrak{S}_b$ , we use the symbol  $S_{ab}$  for the group  $\mathfrak{S}_{ab}$  if it acts by permuting the tensor factors. Since the actions of  $S_{ab}$  and  $\text{GL}(V)$  commute,  $(\bigotimes^{ab} V)_{a \times b}^{\mathfrak{S}_a}$  and  $(\bigotimes^{ab} V)_{b \times a}^{\mathfrak{S}_b}$  are  $S_{ab}$  representations. Recall that the irreducible  $S_{ab}$  representations  $[\lambda]$  are indexed by partitions  $\lambda$  whose Young diagrams have  $ab$  boxes, i.e.,  $|\lambda| = ab$ .

Let  $e_1, \dots, e_a$  denote the standard basis of  $\mathbb{C}^a$  and let  $f_1, \dots, f_b$  denote the standard basis of  $\mathbb{C}^b$ . For  $d \in \mathbb{N}$ ,  $W := \bigotimes^d V$ ,  $1 \leq i \leq a$  and  $1 \leq j \leq b$  we define  $\varphi_{i,j} : W \rightarrow W$  to be the raising operator that projects each  $(\alpha, \beta)$  weight space in  $W$  to the  $(\alpha - e_i, \beta + f_j)$  weight space. For example  $\varphi_{2,1}$  maps the  $((3, 2, 1), (2, 2))$  weight space to the  $((3, 1, 1), (3, 2))$  weight space.

**2.1 Claim.** *All the  $\varphi_{i,j}$  commute.*

We postpone the proof to Section 6.

Define the map  $\varphi_{a \times b} : (\bigotimes^{ab} V)_{a \times b, \emptyset} \rightarrow (\bigotimes^{ab} V)_{\emptyset, b \times a}$  via

$$\varphi_{a \times b} := \varphi_{1,1} \circ \varphi_{1,2} \circ \cdots \circ \varphi_{1,b} \circ \varphi_{2,1} \circ \cdots \circ \varphi_{2,b} \circ \cdots \circ \varphi_{a,b}. \quad (2.2)$$

Note that according to Claim 2.1 the order of the factors in (2.2) does not matter. The restriction of  $\varphi_{a \times b}$  to the linear subspace of  $\mathfrak{S}_a$  invariants shall be denoted by

$$\psi_{a \times b} : (\bigotimes^{ab} V)_{a \times b, \emptyset}^{\mathfrak{S}_a} \rightarrow (\bigotimes^{ab} V)_{\emptyset, b \times a}.$$

It is easy to see that  $\psi_{a \times b}$  actually maps to  $(\bigotimes^{ab} V)_{\emptyset, b \times a}^{\mathfrak{S}_b}$ , but we will omit this detail in the upcoming proofs. Since each  $\varphi_{i,j}$  is  $\mathfrak{S}_{ab}$  equivariant, the map  $\psi_{a \times b}$  is  $\mathfrak{S}_{ab}$  equivariant.

Using Schur-Weyl duality we see that the multiplicity of  $\{\lambda\}$  in  $\text{Sym}^a(\text{Sym}^b U)$  equals the multiplicity of the irreducible  $\mathfrak{S}_{ab}$  representation  $[\lambda]$  in  $(\bigotimes^{ab} V)_{a \times b, \emptyset}^{\mathfrak{S}_a}$  and the multiplicity of  $\{\lambda\}$  in  $\text{Sym}^b(\text{Sym}^a U)$  equals the multiplicity of  $[\lambda]$  in  $(\bigotimes^{ab} V)_{b \times a, \emptyset}^{\mathfrak{S}_b}$ . Moreover, the multiplicity of  $\{\lambda\}$  in the kernel of  $\Psi_{a \times b} : \text{Sym}^a(\text{Sym}^b U) \rightarrow \text{Sym}^b(\text{Sym}^a U)$  is precisely the multiplicity of  $[\lambda]$  in the kernel of  $\psi_{a \times b}$ . Therefore we can phrase McKay's Theorem 1.3 as follows:

**2.3 Theorem** (McKay's main theorem). *If  $\psi_{a \times (b-1)}$  is injective, then  $\psi_{a \times c}$  is injective for all  $c \geq b$ .*

The rest of this paper is devoted to the proof of Theorem 2.3. Using induction it suffices to prove it for the case  $b = c$ . The proof goes by decomposing  $\psi_{a \times b}$  into a composition of two maps (Section 3) and then proving injectivity for the left factor (Section 4) and the right factor (Section 5) separately.

### 3 Decomposition of the canonical map

Let  $a < b$  and set  $B := b - 1$  to simplify notation. We interpret  $\text{GL}_B \subseteq \text{GL}_b$  as  $B \times B$  matrices embedded in the upper left corner of  $b \times b$  matrices with an additional 1 at the lower right corner. Let  $V := \mathbb{C}^a \oplus \mathbb{C}^b$ .

**3.1 Claim.** *Given a tensor power  $W := \bigotimes^d V$ . The composition*

$$\varphi_{1,b} \circ \varphi_{2,b} \circ \cdots \circ \varphi_{a,b} : W \rightarrow W$$

*maps  $\mathfrak{S}_a$ -invariants to  $\mathfrak{S}_a$ -invariants.*

We postpone the proof to Section 6.

Let  $(0^B, a)$  denote the weak composition  $(0, 0, \dots, 0, a) \in \mathbb{N}^b$  that is zero everywhere but in the last entry.

Consider the map  $\psi_{a \times b} : (\bigotimes^{ab} V)_{a \times b, \emptyset}^{\mathfrak{S}_a} \rightarrow (\bigotimes^{ab} V)_{\emptyset, b \times a}$ . According to Claim 3.1 the right factor of

$$\psi_{a \times b} = \underbrace{(\varphi_{1,1} \circ \cdots \circ \varphi_{1,B} \circ \varphi_{2,1} \circ \cdots \circ \varphi_{a,B})}_{\text{left factor}} \circ \underbrace{(\varphi_{1,b} \circ \cdots \circ \varphi_{a,b})}_{\text{right factor}}. \quad (3.2)$$

maps  $\mathfrak{S}_a$  invariants to  $\mathfrak{S}_a$  invariants, so that we can write

$$(\bigotimes^{ab} V)_{a \times b, \emptyset}^{\mathfrak{S}_a} \xrightarrow{\text{right factor}} (\bigotimes^{ab} V)_{a \times B, (0^B, a)}^{\mathfrak{S}_a} \xrightarrow{\text{left factor}} (\bigotimes^{ab} V)_{\emptyset, b \times a}.$$

The left factor is similar to  $\psi_{a \times B}$ , but with a larger domain of definition. The proof idea for Theorem 2.3 is to prove injectivity of both factors independently, where the injectivity of the left factor will follow from the induction hypothesis that  $\psi_{a \times B}$  is injective.

## 4 The left factor

We want to be more precise about the relationship between the left factor of (3.2) and  $\psi_{a \times B}$ . Let  $\mathbb{C}^B \subseteq \mathbb{C}^b$  be embedded as vectors that have a zero as their last component. Let  $V' := \langle e_1, \dots, e_a, f_1, \dots, f_B \rangle = \mathbb{C}^a \oplus \mathbb{C}^B \subseteq V$  be the complement of the 1-dimensional vector space spanned by the basis vector  $f_b$ . We decompose  $\bigotimes^{ab} V$  as follows.  $\bigotimes^{ab} V =$

$$\bigoplus_{Q \subseteq [ab]} \langle \{v_1 \otimes v_2 \otimes \dots \otimes v_{ab} \mid v_i \in V' \text{ if } i \notin Q, v_i = f_b \text{ if } i \in Q\} \rangle, \quad (4.1)$$

where  $[ab] := \{1, 2, \dots, ab\}$  and  $\langle \rangle$  denotes the linear span. We denote by  $\bigotimes_Q^{ab} V$  the summand in (4.1) corresponding to  $Q$ . The weight spaces split as follows:

$$(\bigotimes^{ab} V)_{a \times B, (0^B, a)}^{\mathfrak{S}_a} = \bigoplus_{\substack{Q \subseteq [ab] \\ |Q|=a}} (\bigotimes_Q^{ab} V)_{a \times B, (0^B, a)}^{\mathfrak{S}_a}$$

and

$$(\bigotimes^{ab} V)_{\emptyset, b \times a} = \bigoplus_{\substack{Q \subseteq [ab] \\ |Q|=a}} (\bigotimes_Q^{ab} V)_{\emptyset, b \times a}.$$

As a  $\mathrm{GL}_a \times \mathrm{GL}_B$  representation,  $\bigotimes_Q^{ab} V$  is canonically isomorphic to  $\bigotimes^{ab-|Q|} V'$ . Using this isomorphism, for  $|Q| = a$  we see that the following diagram commutes:

$$\begin{array}{ccc} (\bigotimes_Q^{ab} V)_{a \times B, (0^B, a)}^{\mathfrak{S}_a} & \simeq & (\bigotimes^{aB} V')_{a \times B, \emptyset}^{\mathfrak{S}_a} \\ \psi_{a \times B}^Q \downarrow & & \downarrow \psi_{a \times B} \\ (\bigotimes_Q^{ab} V)_{\emptyset, b \times a} & \simeq & (\bigotimes^{aB} V')_{\emptyset, B \times a} \end{array}$$

where  $\psi_{a \times B}^Q$  is the left factor of (3.2) restricted to  $(\bigotimes_Q^{ab} V)_{a \times B, (0^B, a)}^{\mathfrak{S}_a}$ . Hence we have

$$\psi_{a \times b} = \left( \bigoplus_{\substack{Q \subseteq [ab] \\ |Q|=a}} \psi_{a \times B}^Q \right) \circ (\varphi_{1,b} \circ \varphi_{2,b} \circ \dots \circ \varphi_{a,b}).$$

We see that since  $\psi_{a \times B}$  is injective by induction hypothesis, each  $\psi_{a \times B}^Q$  is injective, and  $\psi_{a \times b}$  is injective as a direct sum of injective maps whose ranges form a direct sum.

## 5 The right factor

To show that the right factor of (3.2) is injective it suffices to show injectivity for the  $a$  factors  $\varphi_{i,b}$ ,  $1 \leq i \leq a$ . Let  $\gamma_i$  denote the weak composition

$$\gamma_i := (\underbrace{B, B, \dots, B}_i, \underbrace{b, b, \dots, b}_{a-i})$$

whose Young diagram has  $ab-i$  boxes (if we flip the rows, then it is a partition). We have  $\gamma_0 = a \times b$  and  $\gamma_a = a \times B$ . With this notation we can write

$$\begin{aligned} & (\otimes^{ab} V)_{a \times b, \emptyset} \\ & \parallel \\ & (\otimes^{ab} V)_{\gamma_0, (0^B, 0)} \xrightarrow{\varphi_{1,b}} (\otimes^{ab} V)_{\gamma_1, (0^B, 1)} \xrightarrow{\varphi_{2,b}} (\otimes^{ab} V)_{\gamma_2, (0^B, 2)} \xrightarrow{\varphi_{3,b}} \dots \xrightarrow{\varphi_{a,b}} (\otimes^{ab} V)_{\gamma_a, (0^B, a)} \\ & \parallel \\ & (\otimes^{ab} V)_{a \times B, (0^B, a)} \end{aligned}$$

Note that formally we are only required to prove the injectivity of this chain  $\varphi_{a,b} \circ \dots \circ \varphi_{1,b}$  restricted to the  $\mathfrak{S}_a$  invariant space  $(\otimes^{ab} V)_{a \times b, \emptyset}^{\mathfrak{S}_a}$ . We ignore the action of  $\mathfrak{S}_a$  and in the following claim we prove the injectivity of each factor of the chain, which finishes the proof of Theorem 2.3.

**5.1 Claim.** *Let  $a < b$  and  $1 \leq i \leq a$ . The map*

$$\varphi_{i,b} : (\otimes^{ab} V)_{\gamma_{i-1}, (0^B, i-1)} \rightarrow (\otimes^{ab} V)_{\gamma_i, (0^B, i)}$$

*is injective.*

*Proof.* Fix  $i$  with  $1 \leq i \leq a$ . Note that the inequality  $i \leq B$  holds, because  $i \leq a < b$ . We will use it later.

We proceed similarly to the proof for the left factor. Let  $V' = \langle e_1, \dots, e_{i-1}, e_{i+1}, \dots, e_a, f_1, f_2, \dots, f_B \rangle$  be the complement of  $\langle e_i, f_b \rangle$ . For a subset  $Q \subseteq [ab]$  we denote by  $\otimes_Q^{ab} V$  the vector space spanned by

$$\{v_1 \otimes \dots \otimes v_{ab} \mid v_i \in V' \text{ if } i \in Q, v_i \in \langle e_i, f_b \rangle \text{ if } i \notin Q\}.$$

The weight spaces split as follows:

$$(\otimes^{ab} V)_{\gamma_{i-1}, (0^B, i-1)} = \bigoplus_{\substack{Q \subseteq [ab] \\ |Q| = ab - (B+i)}} (\otimes_Q^{ab} V)_{\gamma_{i-1}, (0^B, i-1)}$$

and

$$(\otimes^{ab} V)_{\gamma_i, (0^B, i)} = \bigoplus_{\substack{Q \subseteq [ab] \\ |Q| = ab - (B+i)}} (\otimes_Q^{ab} V)_{\gamma_i, (0^B, i)}.$$

We embed  $\text{GL}_2 \subseteq \text{GL}(V)$  to be the  $\text{GL}_2$  that preserves the linear space  $\langle e_i, f_b \rangle$ . As a  $\text{GL}_2$  representation  $\otimes_Q^{ab} V$  is canonically isomorphic to  $\otimes^{ab-|Q|} \mathbb{C}^2$ . Using

this isomorphism, for  $|Q| = ab - (B + i)$  we see that the following diagram commutes:

$$\begin{array}{ccc} (\otimes_Q^{ab} V)_{\gamma_{i-1}, (0^B, i-1)} & \simeq & (\otimes^{B+i} \mathbb{C}^2)_{b, i-1} \\ \varphi_{i,b}^Q \downarrow & & \downarrow \zeta \\ (\otimes_Q^{ab} V)_{\gamma_i, (0^B, i)} & \simeq & (\otimes^{B+i} \mathbb{C}^2)_{B, i} \end{array}$$

where  $\varphi_{i,b}^Q$  is the raising operator  $\varphi_{i,b}$  restricted to the weight space  $(\otimes_Q^{ab} V)_{\gamma_{i-1}, (0^B, i-1)}$ , and  $\zeta : (\otimes^{B+i} \mathbb{C}^2)_{b, i-1} \rightarrow (\otimes^{B+i} \mathbb{C}^2)_{B, i}$  is the canonical  $\mathbf{GL}_2$  raising operator.

It remains to show that  $\zeta$  is injective, because then all  $\varphi_{i,b}^Q$  are injective and hence  $\varphi_{i,b} = \bigoplus_Q \varphi_{i,b}^Q$  is injective as a direct sum of injective maps whose ranges form a direct sum. We will use the fact that  $i \leq B$ .

By Schur-Weyl duality,

$$\otimes^{B+i} \mathbb{C}^2 = \bigoplus_{\lambda} \{\lambda\} \otimes [\lambda]$$

as a  $\mathbf{GL}_2 \times \mathbf{S}_{B+i}$  representation, where  $\lambda$  runs over all partitions with  $B + i$  boxes and at most 2 parts, and  $\{\lambda\}$  denotes the irreducible  $\mathbf{GL}_2$  representation of type  $\lambda$ . The Kostka number determines the dimension of the  $(b, i - 1)$  weight space in  $\{\lambda\}$ : It is 1-dimensional iff  $i - 1 \geq \lambda_2$  and zero otherwise. Therefore as an  $\mathbf{S}_{B+i}$  representation we have

$$(\otimes^{B+i} \mathbb{C}^2)_{(b, i-1)} = \bigoplus_{\substack{\lambda \\ i-1 \geq \lambda_2}} [\lambda].$$

By Schur's lemma, since  $\zeta$  is  $\mathbf{S}_{B+i}$  equivariant, it suffices to check for each partition  $\lambda$  that a single  $(b, i - 1)$  weight vector in the  $\lambda$ -isotypic  $\mathbf{GL}_2$  component is not mapped to zero by  $\zeta$ . We will choose an explicit weight vector as follows. Let  $\{e, f\}$  be the standard basis of  $\mathbb{C}^2$ . For  $B > i - 1 \geq \lambda_2$  we calculate

$$(e \wedge f)^{\lambda_2} \otimes (e^{b-\lambda_2} \cdot f^{i-1-\lambda_2}) \xrightarrow{\zeta} (e \wedge f)^{\lambda_2} \otimes (e^{B-\lambda_2} \cdot f^{i-\lambda_2}) \neq 0,$$

where  $\cdot$  denotes the symmetric product. □

## 6 Appendix: Proofs of the preliminary claims

*Proof of Claim 2.1.* The map  $\varphi_{i,j}$  is defined on basis tensors as follows:

$$\varphi_{i,j}(v_1 \otimes \cdots \otimes v_d) =$$

$$\frac{1}{d} \left( \zeta_{i,j}(v_1) \otimes v_2 \otimes \cdots \otimes v_d + v_1 \otimes \zeta_{i,j}(v_2) \otimes \cdots \otimes v_d + \cdots + v_1 \otimes \cdots \otimes \zeta_{i,j}(v_d) \right),$$

where all  $v_k \in \{e_1, \dots, e_a, f_1, \dots, f_b\}$ , and  $\zeta_{i,j} : V \rightarrow V$  maps  $e_i$  to  $f_j$  and vanishes on all other basis vectors. Since  $\zeta_{i,j} \circ \zeta_{i',j'} = 0$ , for the composition of

maps  $\varphi_{i,j} \circ \varphi_{i',j'}$  we have

$$(\varphi_{i,j} \circ \varphi_{i',j'})(v_1 \otimes \cdots \otimes v_d) = \frac{1}{d(d-1)} \sum_{\substack{p,q=1 \\ p \neq q}}^d v_1 \otimes v_2 \otimes \zeta_{i,j}(v_p) \otimes \cdots \otimes \zeta_{i',j'}(v_q) \otimes \cdots \otimes v_d.$$

This expression is symmetric in  $p$  and  $q$ . Therefore  $\varphi_{i,j} \circ \varphi_{i',j'} = \varphi_{i',j'} \circ \varphi_{i,j}$ .  $\square$

*Proof of Claim 3.1.* The action of  $\mathfrak{S}_a$  permutes the weight spaces. More precisely, for  $\pi \in \mathfrak{S}_a$  and  $w \in W$  we have

$$\pi \varphi_{i,b}(w) = \varphi_{\pi(i),b}(\pi w).$$

Therefore, if we take  $w$  to be  $\mathfrak{S}_a$ -invariant, we see that

$$\pi \left( (\varphi_{1,b} \circ \cdots \circ \varphi_{a,b})(w) \right) = \underbrace{(\varphi_{\pi(1),b} \circ \cdots \circ \varphi_{\pi(a),b})}_{\substack{2.1 \\ \equiv \varphi_{1,b} \circ \cdots \circ \varphi_{a,b}}}(\underbrace{\pi w}_{=w})$$

for every  $\pi \in \mathfrak{S}_a$ , and hence  $(\varphi_{1,b} \circ \cdots \circ \varphi_{a,b})(w)$  is  $\mathfrak{S}_a$ -invariant.  $\square$

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